UNIT 3 SPECIALIST MATHS

COMPLEX NUMBERS

REVISION NOTES FOR YOUR SACS & EXAMS

WRITTEN BY A STUDENT WHO OBTAINED A SCALED STUDY SCORE OF 52.46!

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Chapter 4 Complex Numbers

Exercise 4A Set of Complex Numbers

→ Imaginary number: $i^2 = -1 \rightarrow i = \sqrt{-1}$

: $\sqrt{-a} = i\sqrt{a}$, where $a \in R^+$

→ Set of complex numbers: $C = \{a+bi : a, b, \in R\}$ (where $N \subseteq Z \subseteq Q \subseteq R \subseteq C$)

- Real and imaginary parts
  - for $z = a+bi \rightarrow Re(z) = a$
  - $\rightarrow Im(z) = b$ (not bi!)

· Calc command: Menu → Number (2) → Complex (9)

→ Operations on complex numbers (for $z_1 = a+bi$ and $z_2 = c+di$)

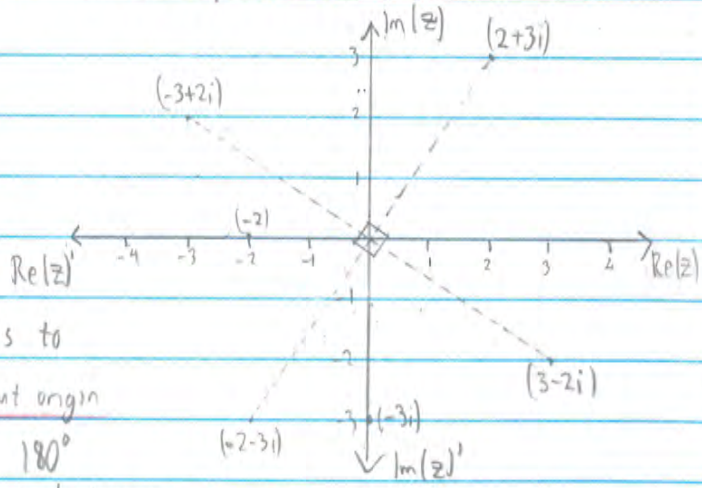
- Addition: $z_1 + z_2 = (a+c) + (b+d)i$
- Subtraction: $z_1 - z_2 = (a-c) + (b-d)i$
- Multiplication: $z_1 \times z_2 = (ac-bd) + (ad+bc)i$

→ Powers of i (for $n \in Z$)

- $i^{4n} = 1$ · $i^{4n+1} = i$ · $i^{4n+2} = -1$ · $i^{4n+3} = -i$

→ Argand diagram (geometric two-dimensional representation of a complex number)

- Geometric representation
  - through addition, subtraction, multiplication by a scalar
- Rotation about origin
  - when multiplied by i → corresponds to rotation of $90°$ anticlockwise about origin
  - when multiplied by -1 → rotates $180°$
  - $\times\, i^n$ → rotation of $\frac{\pi}{2} \times n$ anticlockwise



Exercise 4B Modulus, conjugate and division

→ Modulus (distance of $z = a+bi$ from origin) → length of line

- $|z| = \sqrt{a^2+b^2}$
- $|z_1 z_2| = |z_1||z_2|$ (product)
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ (quotient)
- $|z_1 + z_2| \leq |z_1| + |z_2|$ (triangle inequality)

→ Complex Conjugate (z bar) → mirror image of complex # about x-axis

- If $z = a+bi$, then $\bar{z} = a-bi$
- Properties:
  - $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$
  - $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
  - $\overline{kz} = k\bar{z}$, for $k \in R$
  - $z\bar{z} = |z|^2 = a^2+b^2$
  - $z+\bar{z} = 2Re(z)$
  - $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$

→ Division of complex numbers (Rationalise denominator with complex conjugate)

- e.g. 1. $\frac{8-i}{2+3i} = \frac{(8-i)(2-3i)}{(2+3i)(2-3i)} = \frac{16-24i-2i+3i^2}{2^2-(3i)^2} = \frac{16-3-26i}{4+9} = \frac{13-26i}{13}$

  $\therefore = \boxed{1-2i}$

- e.g. 2. $\frac{(1+2i)^2}{i(1+3i)} = \frac{1+4i-4}{-3+i} = \frac{(-3+4i)(-3-i)}{(-3+i)(-3-i)} = \frac{9+3i-12i+4}{(-3)^2-i^2} = \boxed{\frac{13-9i}{10}}$

- e.g. 3. $\frac{1}{i} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{i^2} = \boxed{-i}$

→ Application: e.g. $z = x+iy$, $C = e+id$

Show that $(z-C)(\bar{z}-\bar{C}) = r^2$ is eqn of a circle, centre (e,d), radius r.

$(x+iy-(e+id))(x-iy-(e-id)) = r^2$

$[(x-e)+i(y-d)][(x-e)-i(y-d)] = r^2$

$(x-e)^2 - (i(y-d))^2 = r^2$

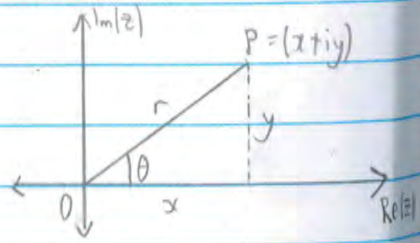
$\boxed{(x-e)^2 + (y-d)^2 = r^2}$

Circle, centre (e,d), radius r

Exercise 4C Modulus-Argument form of a complex number

→ Polar form → $z = r\operatorname{cis}\theta = r(\cos\theta + i\sin\theta)$

(from Cartesian form $z = x+yi$)



- Modulus: $r = \sqrt{x^2+y^2}$ - denoted by $|z|$
- Argument: angle θ from positive direction of x-axis
  - $Arg(z) = \theta \in [0,\pi]$ → Quadrant 1 or 2
  - $\in (-\pi, 0)$ → Quadrant 3 or 4

→ Steps from Cartesian to Polar

① $r = \sqrt{x^2+y^2}$

② $\tan\theta = \frac{y}{x}$ → $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

③ $z = r\operatorname{cis}\theta$ ($= re^{i\theta}$ on CAS)

- Calc: $ctp(a,b) = [\sqrt{a^2+b^2} \ \ angle(a+bi)]$

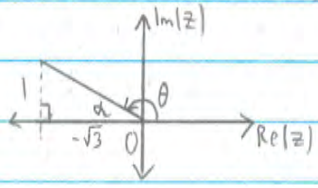
- e.g. Express $-\sqrt{3}+i$ in polar form as $z = r\operatorname{cis}\theta$

① $r = \sqrt{(\sqrt{3})^2 + 1^2} = 2$

② $\alpha = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$\theta = \frac{5\pi}{6}$

③ $\boxed{z = 2\operatorname{cis}\left(\frac{5\pi}{6}\right)}$



→ Steps from Polar to Cartesian

$z = r\operatorname{cis}\theta = \underbrace{r\cos\theta}_{x} + i\underbrace{r\sin\theta}_{y}$

- Calc: ptc(r, θ) = r·(cos(θ)+sin(θ)i)

→ Complex conjugate in polar form.

- If $z = r\operatorname{cis}\theta$, then $\bar{z} = r\operatorname{cis}(-\theta)$

Exercise 4D Basic operations on complex numbers in modulus-argument form

→ Addition and subtraction - must be converted to cartesian form first

- e.g. Find $2\operatorname{cis}\left(\frac{2\pi}{3}\right) - 3\operatorname{cis}\left(-\frac{\pi}{3}\right)$.

$2\operatorname{cis}\left(\frac{2\pi}{3}\right) = 2\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) = 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -1 + \sqrt{3}i$

$3\operatorname{cis}\left(-\frac{\pi}{3}\right) = 3\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right) = 3\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \frac{3}{2} - \frac{3\sqrt{3}}{2}i$

$\therefore (-1+\sqrt{3}i) - \left(\frac{3}{2} - \frac{3\sqrt{3}}{2}i\right) = \boxed{-\frac{5}{2} + \frac{5\sqrt{3}}{2}i}$

→ Multiplication by a scalar

- Positive: If $k \in R^+$, then $\operatorname{Arg}(kz) = \operatorname{Arg}(z)$
- Negative: If $k \in R^-$, then

$$\operatorname{Arg}(kz) = \begin{cases} \operatorname{Arg}(z) - \pi, & 0 < \operatorname{Arg}(z) \leq \pi \\ \operatorname{Arg}(z) + \pi, & -\pi < \operatorname{Arg}(z) \leq 0 \end{cases}$$

→ Multiplication of complex numbers

- $z_1 = r_1\operatorname{cis}(\theta_1)$, $z_2 = r_2\operatorname{cis}(\theta_2)$
- $z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$

→ Division of complex numbers

- $z_1 = r_1\operatorname{cis}(\theta_1)$, $z_2 = r_2\operatorname{cis}(\theta_2)$
- $\frac{z_1}{z_2} = \frac{r_1}{r_2}\operatorname{cis}(\theta_1 - \theta_2)$

→ De Moivre's Theorem: $(r\operatorname{cis}(\theta))^n = r^n\operatorname{cis}(n\theta)$, $n \in Z$

- e.g. 1. Simplify $\frac{(1-i)^3}{(1+\sqrt{3}i)^4}$.

→ $1 - i = \sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$, $1 + \sqrt{3}i = 2\operatorname{cis}\left(\frac{\pi}{3}\right)$

$$\frac{(\sqrt{2}\,\text{cis}(-\frac{\pi}{4}))^3}{(2\,\text{cis}(\frac{\pi}{3}))^4} = \frac{2\sqrt{2}\,\text{cis}(-\frac{3\pi}{4})}{2^4\,\text{cis}(\frac{4\pi}{3})}$$
$$= \frac{\sqrt{2}}{8}\text{cis}\left(-\frac{9\pi}{12} - \frac{16\pi}{12}\right)$$
$$= \frac{\sqrt{2}}{8}\text{cis}\left(-\frac{25\pi}{12}\right) = \boxed{\frac{\sqrt{2}}{8}\text{cis}\left(-\frac{\pi}{12}\right)}$$

e.g. 2. Simplify $(-1+i)^5\left(\frac{1}{2}\text{cis}\left(\frac{\pi}{4}\right)\right)^3$

$$\rightarrow \left(\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)\right)^5\left(\frac{1}{8}\text{cis}\left(\frac{3\pi}{4}\right)\right) = 4\sqrt{2}\,\text{cis}\left(\frac{15\pi}{4}\right) \times \frac{1}{8}\text{cis}\left(\frac{3\pi}{4}\right)$$
$$= \frac{4\sqrt{2}}{8}\text{cis}\left(\frac{15\pi}{4} + \frac{3\pi}{4}\right)$$
$$= \frac{\sqrt{2}}{2}\text{cis}\left(\frac{9\pi}{2}\right)$$
$$= \frac{\sqrt{2}}{2}\text{cis}\left(\frac{\pi}{2}\right) = \boxed{\frac{\sqrt{2}}{2}i}$$

Exercise 4E Solving quadratic equations over complex numbers

e.g. 1. Solve $2z^2 - 2(3-i)z + 4 - 3i = 0$ for $z \in C$.

$$\rightarrow 2\left(z^2 - (3-i)z + \frac{4-3i}{2}\right) = 0$$
$$2\left[z^2 - (3-i)z + \frac{(3-i)^2}{4} - \frac{(3-i)^2}{4} + \frac{4-3i}{2}\right] = 0$$
$$2\left[z^2 - (3-i)z + \frac{(3-i)^2}{4}\right] - \frac{(3-i)^2}{2} + 4 - 3i = 0$$
$$2\left(z - \frac{(3-i)}{2}\right)^2 - (4-3i) + (4-3i) = 0$$
$$2\left(z - \frac{(3-i)}{2}\right)^2 = 0$$
$$\rightarrow \boxed{z = \frac{3-i}{2}}$$

e.g. 2. Solve $z^2 + (1+2i)z + (-1+i) = 0$ for $z \in C$

$$\rightarrow z^2 + (1+2i)z + \frac{(1+2i)^2}{4} - \frac{(1+2i)^2}{4} + (-1+i) = 0$$
$$\left(z + \frac{(1+2i)}{2}\right)^2 + \frac{4i - 4 - (1+4i+4i^2)}{4} = 0$$
$$\left(z + \frac{(1+2i)}{2}\right)^2 + \frac{-4-1+4}{4} = 0$$
$$\left(z + \frac{(1+2i)}{2}\right)^2 - \frac{1}{4} = 0$$
$$\left(z + \frac{(1+2i)}{2}\right)^2 = \frac{1}{4}$$
$$z + \frac{1+2i}{2} = -\frac{1}{2} \quad \text{or} \quad z + \frac{1+2i}{2} = \frac{1}{2}$$
$$\therefore \boxed{z = -i \quad \text{or} \quad z = -1-i}$$

Alternative Method: $z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ for $az^2 + bz + c = 0$

Exercise 4F Solving polynomial equations over complex numbers

→ Fundamental theorem of algebra

- A polynomial of degree n can be factorised into n linear factors in $\mathbb{C}$.
  i.e. $P(z) = a_n(z-a_1)(z-a_2)(z-a_3)\dots(z-a_n)$, where $a_1, a_2, a_3, \dots, a_n \in \mathbb{C}$

→ Conjugate root theorem

- Let $P(z)$ be a polynomial with real coefficients. If $a+bi$ is a solution of the equation $P(z)=0$, with a and b real numbers, then the complex conjugate $a-bi$ is also a solution.

→ Calc commands: Menu → Algebra (3) → Complex (C) → Factor (2)

→ Examples

- e.g. 1. Factorise $z^3 - 2iz^2 - 6z + 12i$.

$$\to z^2(z-2i) - 6(z-2i)$$
$$= (z^2-6)(z-2i)$$
$$= \boxed{(z+\sqrt{6})(z-\sqrt{6})(z-2i)}$$

- e.g. 2. Let $P(z) = 2z^3 + 9z^2 + 14z + 5$.
  (a) Use the factor theorem to show that $z+2-i$ is a linear factor of $P(z)$.
  (b) Hence find all the linear factors of $P(z)$ over $\mathbb{C}$.

(a) If $(z+2-i)$ is a factor, $P(-2+i) = 0$

$$P(-2+i) = 2(-2+i)^3 + 9(-2+i)^2 + 14(-2+i) + 5$$
$$= 2(-2+11i) + 9(3-4i) + 14i - 28 + 5$$
$$= -4 + 22i + 27 - 36i + 14i - 28 + 5$$
$$\therefore = 0 \text{ (QED)}$$

→ Hence, $(z+2-i)$ is a factor (as required)

(b) $(z+2+i)$ is also a factor.

Hence, $P(z) = (z+2-i)(z+2+i)(2z-\alpha)$
$$= ((z+2)^2 - i^2)(2z-\alpha)$$
$$= (z^2+4z+5)(2z-\alpha)$$
$$= 2z^3 + (8-a)z^2 + (10-4a)z^2 - 5a$$

→ Equating coefficients, $\alpha = -1$

$\therefore$ Linear factors: $\boxed{(z+2-i), (z+2+i), (2z+1)}$

e.g. 3. Factorise $z^6 - 64$ into linear factors over $\mathbb{C}$

$\rightarrow (z^3-8)(z^3+8) = (z-2)(z^2+2z+4)(z+2)(z^2-2z+4)$

$= (z-2)(z^2+2z+1+3)(z+2)(z^2-2z+1+3)$

$= (z-2)((z+1)^2-3i^2)(z+2)((z-1)^2-3i^2)$

$= \boxed{(z-2)(z+1+\sqrt{3}i)(z+1-\sqrt{3}i)(z+2)(z-1+\sqrt{3}i)(z-1-\sqrt{3}i)}$

e.g. 4. For the polynomial $P(z) = az^4 + az^2 - 2z + d$, where a and d are real numbers.

(a) Evaluate $P(1+i)$. (b) Given than $P(1+i)=0$, find values of a and d.

(c) Show that $P(z)$ can be written as product of two quadratic factors with real coefficients, and hence solve the equation $P(z)=0$.

(a) $P(1+i) = a(1+i)^4 + a(1+i)^2 - 2(1+i) + d$

$= \boxed{(-4a+d-2)+2(a-1)i}$

(b) Equating coefficients, $-4a+d-2 = 0 \cdots ①$

$2(a-1) = 0 \cdots ②$

$\therefore \boxed{a=1, d=6}$

(c) If $P(1+i)=0$, then also $P(1-i)=0$

$(z-1+i)(z-1-i) = (z-1)^2+1 = z^2-2z+2$

$$\begin{array}{r} z^2+2z+3 \\ z^2-2z+2 \,\overline{)\, z^4+0z^3+z^2-2z+6} \\ \underline{z^4-2z^3+2z^2} \\ 2z^3-z^2-2z \\ \underline{2z^3-4z^2+4z} \\ 3z^2-6z+6 \\ \underline{3z^2-6z+6} \\ 0 \end{array}$$

$\rightarrow P(z) = (z^2-2z+2)(z^2+2z+3)$

$\rightarrow$ $\boxed{\text{Product of two quadratic factors (QED)}}$

$P(z) = 0 \rightarrow (z^2-2z+1+1)(z^2+2z+1+2) = 0$

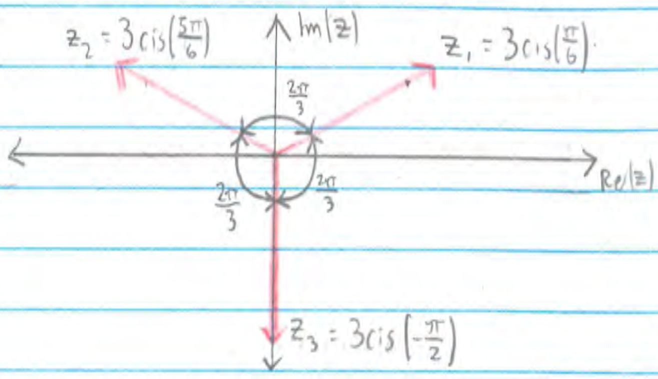
$((z-1)^2-i^2)((z+1)^2-2i^2) = 0$

$(z-1+i)(z-1-i)(z+1+\sqrt{2}i)(z+1-\sqrt{2}i) = 0$

$\therefore \boxed{z = 1 \pm i, -1 \pm \sqrt{2}i}$

Exercise 4G Using De Moivre's theorem to solve equations

→ e.g. 1. Solve $z^3 = 27i$ over $z \in \mathbb{C}$



$z^3 = 27\operatorname{cis}\left(\frac{\pi}{2} + 2k\pi\right),\ k \in \mathbb{Z}$

$z = 3\operatorname{cis}\left(\frac{\pi}{6} + \frac{2}{3}k\pi\right),\ k \in \mathbb{Z}$

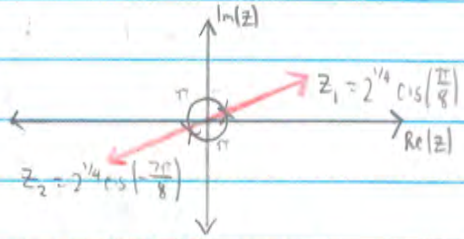
If $k=0$, $z_1 = 3\operatorname{cis}\left(\frac{\pi}{6}\right)$

If $k=1$, $z_2 = 3\operatorname{cis}\left(\frac{5\pi}{6}\right)$

If $k=2$, $z_3 = 3\operatorname{cis}\left(\frac{3\pi}{2}\right) = 3\operatorname{cis}\left(-\frac{\pi}{2}\right)$

$\boxed{z = 3\operatorname{cis}\left(\frac{\pi}{6}\right), 3\operatorname{cis}\left(\frac{5\pi}{6}\right), 3\operatorname{cis}\left(-\frac{\pi}{2}\right)}$

→ e.g. 2. Solve $z^2 = 1+i$ over $z \in \mathbb{C}$



$z^2 = \sqrt{2}\operatorname{cis}\left(\frac{\pi}{4} + 2k\pi\right),\ k \in \mathbb{Z}$

$z = 2^{1/4}\operatorname{cis}\left(\frac{\pi}{8} + k\pi\right),\ k \in \mathbb{Z}$

If $k=0$, $z_1 = \boxed{2^{1/4}\operatorname{cis}\left(\frac{\pi}{8}\right)}$

If $k=1$, $z_2 = 2^{1/4}\operatorname{cis}\left(\frac{9\pi}{8}\right) = \boxed{2^{1/4}\operatorname{cis}\left(-\frac{7\pi}{8}\right)}$

→ e.g. 3. Solve $z^2 = -15-8i$ using $z = a+bi$, where $a, b \in \mathbb{R}$.
Hence, factorise $z^2 + 15 + 8i$

→ $z^2 = (a+bi)^2 = a^2 + 2abi + b^2i^2$

$= (a^2 - b^2) + 2abi = -15 - 8i$

$a^2 - b^2 = -15 \ \cdots (1)$, $2ab = -8 \ \cdots (2)$

From (2), $b = -\frac{4}{a}$

Sub into (1), $a^2 - \left(-\frac{4}{a}\right)^2 = -15$

$a^4 - 16 = -15a^2$

$a^4 + 15a^2 - 16 = 0$

$(a^2 + 16)(a^2 - 1) = 0$

$(a+1)(a-1)(a+4i)(a-4i) = 0$

$a = \pm 1$ (accept) or $a = \pm 4i$ (reject) (since $a \in \mathbb{R}$)

If $a = +1$, $b = -4$

If $a = -1$, $b = +4$

Hence $\boxed{z = 1-4i \text{ or } z = -1+4i}$

→ $z^2 + 15 + 8i = (z - 1 + 4i)(z + 1 - 4i)$

→ Steps: (1) Express z in form $z = r\operatorname{cis}(\theta + 2k\pi)$, $k = 0, \pm 1, \pm 2, \ldots$ and $\theta = \operatorname{Arg}(z)$

(2) Apply De Moivre's Theorem: $z^{1/n} = r^{1/n}\operatorname{cis}\left(\frac{\theta + 2k\pi}{n}\right)$, $k = 0, \pm 1, \pm 2, \ldots$

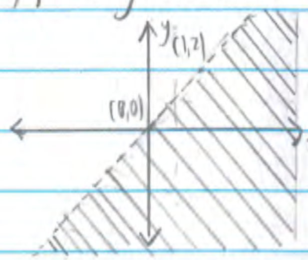
(3) Substitute n consecutive values of k → $k = 0, 1, 2, 3, \ldots, n-1$

(4) Convert polar form into Cartesian (if necessary)

Exercise 4H Sketching subsets of the complex plane

→ Loci (set of all points which satisfy a given condition geometrically)

- e.g. $\{z : \mathrm{Im}(z) < 2\mathrm{Re}(z)\} \to y < 2x$



- Vertical: $\mathrm{Re}(z - z_1) = a$
- Horizontal: $\mathrm{Im}(z - z_1) = a$
- Straight line (Gradient $\neq 0$)
  → $a\,\mathrm{Re}(z - z_1) \pm \mathrm{Im}(z - z_2) = b$

→ Examples of subsets

- e.g. 1. On an Argand diagram, sketch the subset S of the complex plane, where
  $S = \{z : |z - (1+\sqrt{3}i)| = 2\}$ (Circle)

  → Definition: modulus of $z - (1+\sqrt{3}i)$, or distance of z from $(1+\sqrt{3}i)$

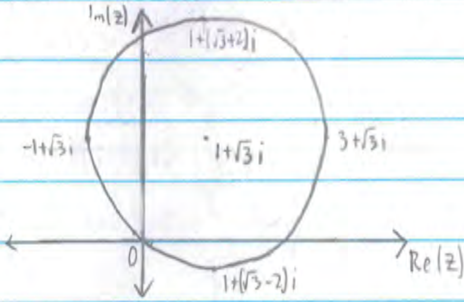
  → $|z - (1+\sqrt{3}i)| = 2$

  $|x + iy - (1+\sqrt{3}i)| = 2$

  $|(x-1) + (y-\sqrt{3})i| = 2$

  $\sqrt{(x-1)^2 + (y-\sqrt{3})^2} = 2$

  $\boxed{(x-1)^2 + (y-\sqrt{3})^2 = 4}$



- e.g. 2. On an Argand diagram, sketch the curve $\left|\frac{z-1-i}{z}\right| = 1$

  → $|z - 1 - i| = |z|$

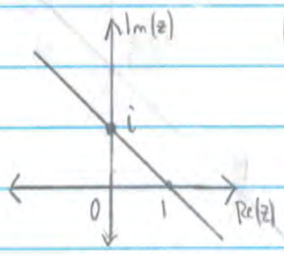
  $|x + iy - 1 - i| = |x + iy|$

  $\sqrt{(x-1)^2 + (y-1)^2} = \sqrt{x^2 + y^2}$

  $x^2 - 2x + 1 + y^2 - 2y + 1 = x^2 + y^2$

  $2x + 2y = 2$

  → $\boxed{y = -x + 1}$

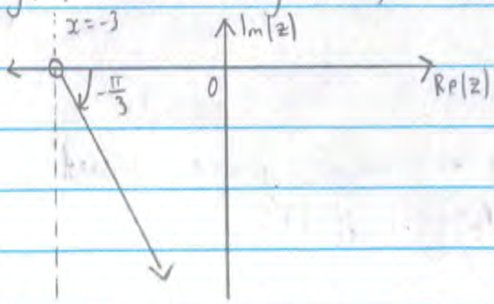


→ $|z - (a+bi)| = |z - (c+di)|$

(Straight line → perpendicular bisector of line segment joining z_1 and z_2)

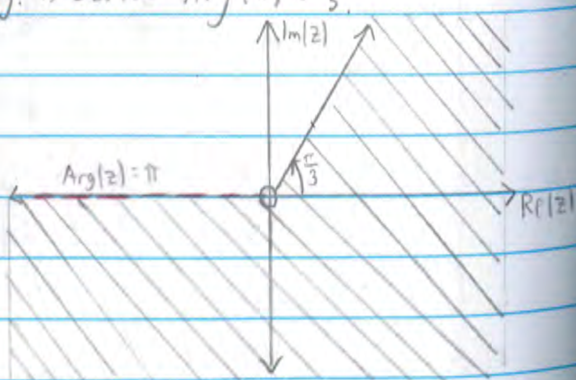
- set of values equidistance btw (a,b) and (c,d)

- e.g. 3. Sketch $\mathrm{Arg}(z+3) = -\frac{\pi}{3}$



N.B. open dot required since angle can't form at origin itself

- e.g. 4. Sketch $\mathrm{Arg}(z) \leq \frac{\pi}{3}$



· e.g. 5. Describe the locus defined by $|z+3| = 2|z-i|$

$\rightarrow |x+iy+3| = 2|x+iy-i|$

$|(x+3)+iy| = 2|x+i(y-1)|$

$\sqrt{(x+3)^2+y^2} = 2\sqrt{x^2+(y-1)^2}$

$(x+3)^2+y^2 = 4(x^2+(y-1)^2)$

$x^2+6x+9+y^2 = 4(x^2+y^2-2y+1)$

$= 4x^2+4y^2-8y+4$

$3x^2-6x+3y^2-8y-5=0$

$3(x^2-2x)+3(y^2-\frac{8}{3}y)-5=0$

$3(x^2-2x+1-1)+3(y^2-\frac{8}{3}y+\frac{16}{9}-\frac{16}{9})-5=0$

$3(x^2-2x+1)-3+3(y^2-\frac{8}{3}y+\frac{16}{9})-\frac{16}{3}-5=0$

$3(x-1)^2+3(y-\frac{4}{3})^2-\frac{40}{3}=0$

$\therefore \boxed{(x-1)^2-(y-\frac{4}{3})^2=\frac{40}{9}}$

Circle, centre $(1, \frac{4}{3})$, radius $\frac{2\sqrt{10}}{3}$

$\rightarrow |z-(a+bi)| = n|z-(c+di)|$

circle (where $n \in [0,1) \cup (1,\infty)$)

$\rightarrow |z-(a+bi)| \pm |z-(c+di)| = k$

line (where $k = \sqrt{(c-a)^2+(d-b)^2}$)

ellipse (where $k > \sqrt{(c-a)^2+(d-b)^2}$)

hyperbola (where $k < \sqrt{(c-a)^2+(d-b)^2}$)

· e.g. 6. Find the locus defined by $|z+2|-|z-2|=3$

$\rightarrow |x+iy-2|-|x+iy+2|=3$

$\sqrt{(x-2)^2+y^2}-\sqrt{(x+2)^2+y^2}=3$

$\sqrt{(x-2)^2+y^2} = 3+\sqrt{(x+2)^2+y^2}$

$(x-2)^2+y^2 = 9+6\sqrt{(x+2)^2+y^2}+(x+2)^2+y^2$

$x^2-4x+4+y^2 = 9+6\sqrt{(x+2)^2+y^2}+x^2+4x+4+y^2$

$-8x-9 = 6\sqrt{(x+2)^2+y^2}$

N.B. $-8x-9>0 \rightarrow x<-\frac{9}{8}$

$\rightarrow 64x^2+144x+81 = 36(x^2+4x+4+y^2)$

$28x^2-36y^2=63$

$\therefore \boxed{\frac{4x^2}{9}-\frac{4y^2}{7}=1}$

Hyperbola, asymptotes $y=\pm\frac{\sqrt{7}}{3}x$, and $x \leq -\frac{3}{2}$

· e.g. 7. Find the set of points in the complex plane defined by $|z|=|z-3|$

$\rightarrow |x+iy| = |x+iy-3| \rightarrow \sqrt{x^2+y^2} = \sqrt{(x-3)^2+y^2}$

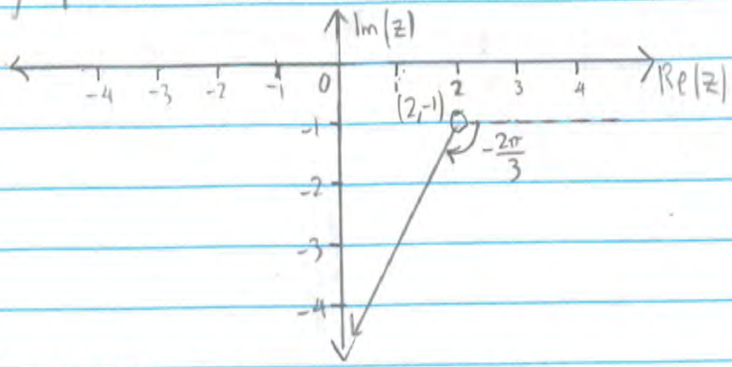
$x^2+y^2 = x^2-6x+9+y^2$

$6x=9 \rightarrow x=\frac{3}{2}$

$* \rightarrow \boxed{\operatorname{Re}(z)=\frac{3}{2}}$

e.g. 8. Consider the relation $Arg(z-2+i) = -\frac{2\pi}{3}$ where $z \in \mathbb{C}$

(a) Draw a graph of this relation.



$Arg(z-(a+bi)) = \theta$

(b) Find the cartesian equation of this relation.

$\rightarrow m = \tan\left(-\frac{2\pi}{3}\right) = \sqrt{3}$

$y-(-1) = \sqrt{3}(x-2)$

$\therefore \boxed{y+1 = \sqrt{3}(x-2),\ x<2}$ (Always state domain!)

Ch 5 (ER) Q2b

e.g. 9. Let S and T be subsets of complex plane given by

$S = \{z : \sqrt{2} \le |z| \le 3 \text{ and } \frac{\pi}{2} < Arg(z) \le \frac{3\pi}{4}\}$

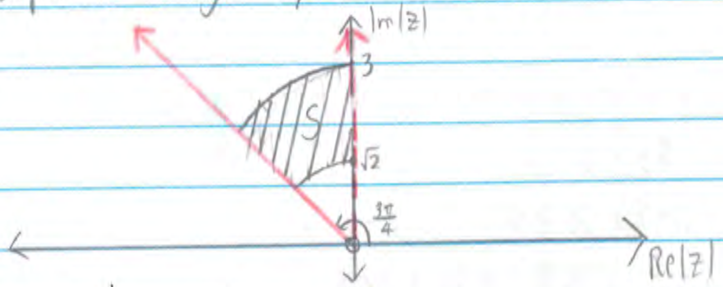
$T = \{z : z\bar{z} + 2Re(iz) \le 0\}$

(a) Sketch S on an Argand diagram

(b) Find $\{z : z \in S \text{ and } z = x+yi \text{ where } x \text{ and } y \text{ are integers}\}$.

(c) On a separate diagram, sketch $S \cap T$.

(a)



(b) Note that when $x<0$, $y>0$.

As $\sqrt{2} \le |z| \le 3$, $\sqrt{2} \le |x+iy| \le 3$

$\sqrt{2} \le \sqrt{x^2+y^2} \le 3$

$2 \le x^2+y^2 \le 9$

As $\frac{\pi}{2} < Arg(z) \le \frac{3\pi}{4}$, $-\infty < \tan(Arg(z)) \le -1$

$\frac{y}{x} \le -1 \rightarrow y \ge -x \ (x<0)$

Solutions:

$z_1 = -1+i$
$z_2 = -1+2i$
$z_3 = -2+2i$

See test and check table next page.

$x\ (<0)$	$y\ (\geq -x)$	x^2+y^2	$2 \leq x^2+y^2 \leq 9$	z
-1	1	2	Y	$-1+i$
-1	2	5	Y	$-1+2i$
-1	3	10	N	
-2	2	8	Y	$-2+2i$
-2	3	13	N	
-3	3	18	N	

(c) $T: (x+iy)(x-iy) + 2\,\mathrm{Re}(i(x+iy)) \leq 0$

$x^2 - i^2y^2 + 2\mathrm{Re}(ix + i^2y) \leq 0$

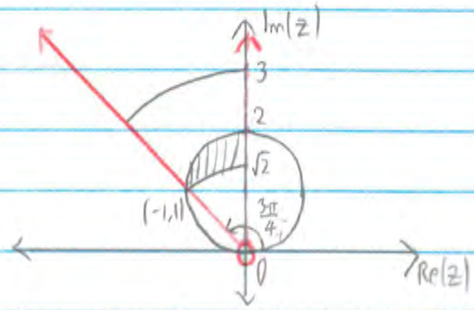
$x^2 + y^2 + 2\mathrm{Re}(-y + ix) \leq 0$

$x^2 + y^2 + 2(-y) \leq 0$

$x^2 + y^2 - 2y \leq 0$

$x^2 + y^2 - 2y + 1 - 1 \leq 0$

$x^2 + (y-1)^2 \leq 1$



- e.g. 10. Consider the circle defined by relation $4(2z+3-2i)(2\bar{z}+3+2i) = 25$, $z \in C$.

Express in form $|z-\alpha| = \sigma$ where $\alpha \in C$ and $\sigma \in R$

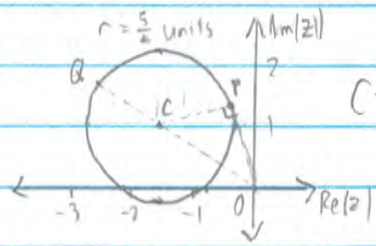
$\rightarrow (2z+3-2i)(\overline{2z+3-2i}) = \frac{25}{4} \rightarrow |2z+3-2i|^2 = \frac{25}{4}$

$4\left|z+\frac{3}{2}-i\right|^2 = \frac{25}{4} \rightarrow \left|z+\frac{3}{2}-i\right|^2 = \frac{25}{16}$

$\rightarrow \boxed{\left|z - \left(-\frac{3}{2}+i\right)\right| = \frac{5}{4}}$

Largest possible modulus: $OQ = OC + CQ$

$(CQ = \text{radius}) \qquad = \sqrt{\left(-\frac{3}{2}\right)^2 + 1^2} + \frac{5}{4} = \frac{2\sqrt{13}+5}{4}$



$C = \left(-\frac{3}{2}, 1\right)$

Smallest positive argument (Arg): $(\overrightarrow{OP} = u\underset{\sim}{i} + v\underset{\sim}{j})$

Modulus: $OP^2 + \left(\frac{5}{4}\right)^2 = \frac{13}{4} \rightarrow OP = \frac{3\sqrt{3}}{4}$

$\overrightarrow{OP} \cdot \underset{\sim}{i} = |\overrightarrow{OP}||\underset{\sim}{i}|\cos(\theta) = OP\cos(\theta)$

$u = OP\cos(\theta) = \frac{3\sqrt{3}}{4}\cos(\theta) \rightarrow \cos(\theta) = \frac{4\sqrt{3}}{9}u$

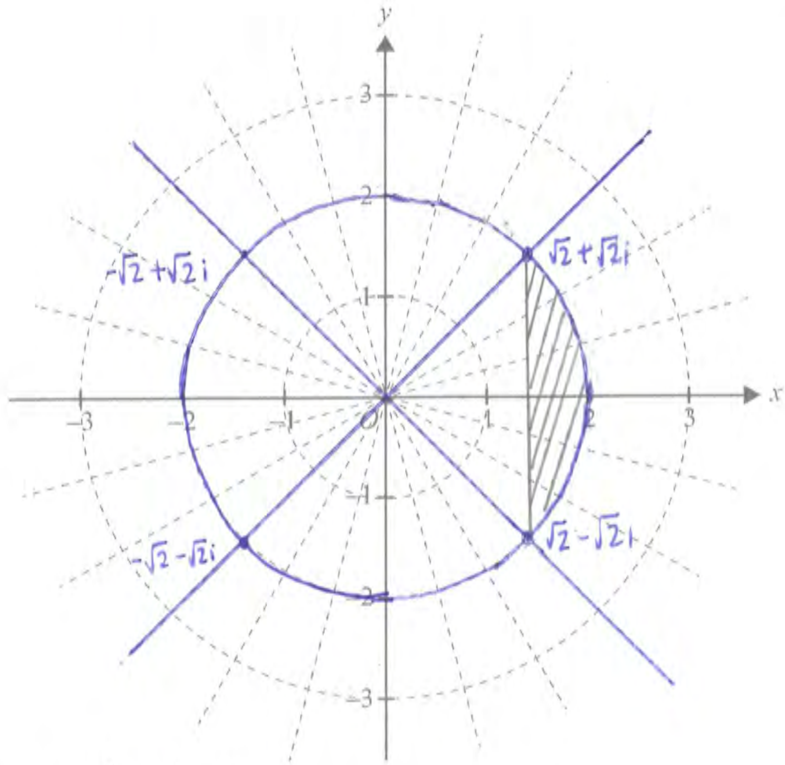
CAS skills (Graph)

1. Grid: Menu → View(2) → Grid(6) → Lined Grid(3)
2. Perp bisect: Menu → Geometry(8) → Construction(4) → Perpendicular Bisector(3)
3. Reveal eqn: Curser on line → Ctrl + Menu → Coordinates and Equation(7)

VCAA 2013 Exam 2 Q2

Question 2 (12 marks)

a. On the Argand diagram below, sketch $\{z : z\bar{z} = 4,\ z \in C\}$ **and** sketch $\{z : |z + \bar{z}| = |z - \bar{z}|,\ z \in C\}$. 3 marks



$z\bar{z} = 4 \to x^2 + y^2 = 4$

$|z + \bar{z}| = |z - \bar{z}| \to |y| = |x|$

b. Find all elements of $\{z : z\bar{z} = 4,\ z \in C\} \cap \{z : |z + \bar{z}| = |z - \bar{z}|,\ z \in C\}$, expressing your answer(s) in the form $a + ib$. 3 marks

(b) In 1st quadrant, $x = 2\cos\left(\frac{\pi}{4}\right) = \sqrt{2}$, $y = 2\sin\left(\frac{\pi}{4}\right) = \sqrt{2}$

$\therefore z_1 = \sqrt{2} + \sqrt{2}i,\ z_2 = -\sqrt{2} + \sqrt{2}i,\ z_3 = -\sqrt{2} - \sqrt{2}i,\ z_4 = \sqrt{2} - \sqrt{2}i$

(Solve $y = \pm x$ and $x^2 + y^2 = 4$ on CAS)

c. One of the roots of the equation $z^4 + 16 = 0$ is $z = \sqrt{2} + i\sqrt{2}$.
Write down the other roots in cartesian form.
Plot and label all of these roots on the Argand diagram provided in **part a.** 2 marks

$z = -\sqrt{2} + \sqrt{2}i,\ -\sqrt{2} - \sqrt{2}i,\ \sqrt{2} - \sqrt{2}i$

d. Express $z^4 + 16$ as the product of four linear factors in terms of z. 1 mark

$(z+2+\sqrt{2}i)(z+\sqrt{2}-\sqrt{2}i)(z-\sqrt{2}+\sqrt{2}i)(z-\sqrt{2}-\sqrt{2}i)$

e. On the Argand diagram provided in **part a.**, shade the region defined by

$$\{z:|z|\le 2,\ z\in C\}\cap\{z:\mathrm{Re}(z)\ge\sqrt{2},\ z\in C\}$$

1 mark

(Shaded in grey pencil)

f. Find the area of the shaded region in **part e.** 2 marks

(f) Area $= \frac{1}{4}\pi r^2 - \frac{1}{2}r^2$ (Area of triangle $= \frac{1}{2}bc\sin(A)$)

$= \frac{1}{4}\pi(2)^2 - \frac{1}{2}(2)^2 = \boxed{(\pi - 2)\text{ units}^2}$ (Area of segment $= \frac{1}{2}r^2(\theta - \sin(\theta))$)

VCAA 2008 Exam 1 Q10

Let $w = 1 + ai$ where a is a real constant.

(a) Show that $|w^3| = (1+a^2)^{3/2}$ (1 mark)

(b) Find the values of a for which $|w^3| = 8$. (1 mark)

(c) Let $p(z) = z^3 + bz^2 + cz + d$, where b, c and d are non-zero real constants. If $p(z) = 0$ for $z = w$ and all roots of $p(z) = 0$ satisfy $|z^3| = 8$, find the values of b, c and d and show that these are the only possible values. (4 marks)

(a) $|w| = (1+a^2)^{1/2}$

$\rightarrow |w^3| = |w|^3 = ((1+a^2)^{1/2})^3 = \boxed{(1+a^2)^{3/2}}$ (as req)

(b) $(1+a^2)^{3/2} = 8 \rightarrow 1 + a^2 = 4$

$\rightarrow a^2 = 3 \rightarrow \boxed{a = \pm\sqrt{3}}$

(c) $|z^3| = 8 \rightarrow z = 1+\sqrt{3}i,\ z = 1-\sqrt{3}i,\ z = -2,\ z = 2$ (from part (b))

There must be 3 roots of $P(z) = 0$

$p(z) = (z-1-\sqrt{3}i)(z-1+\sqrt{3}i)(z-m)$, $m = -2$ or 2

$p(z) = [(z-1)^2 - (\sqrt{3}i)^2](z-m) = (z^2 - 2z + 4)(z-m)$

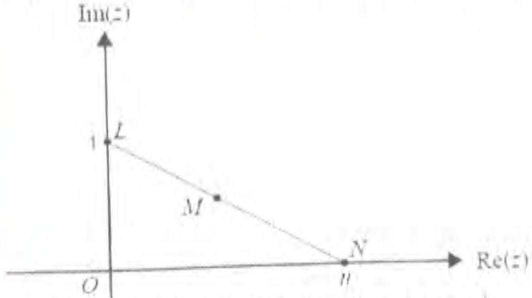
$= z^3 - (m+2)z^2 + (2m+4)z - 4m$

$\rightarrow m + 2 \ne 0,\ 2m + 4 \ne 0 \rightarrow m \ne -2,\ m \ne 0 \rightarrow m = 2$ is valid solution (QED)

$\therefore \boxed{b = -4,\ c = 8,\ d = -8}$

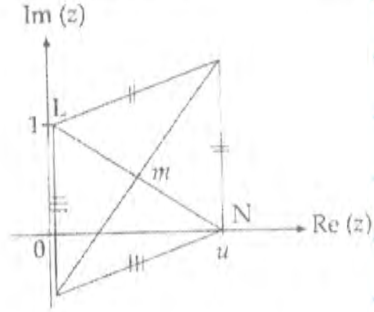
VCAA 2002 Exam 2 Extension Question 5 (total 10 marks)

In the Argand diagram below, L is the point i, N is the point $u + 0i$, and M is the midpoint of LN.



5ai) When z lies on LN it is equal distance from L and N that means it is on the midpoint, m.

When z lies on either side of the LN it is equidistant from L and N, and forms an isosceles triangle. The line drawn from the vertex of an isosceles triangle bisecting the base of the triangle is perpendicular to the base.



Let $z = x + yi$ satisfy $|z - i| = |z - u|$.

a) i) Why does z lie on the perpendicular bisector of LN? (1 mark)

ii) Show that $2y = 2ux - u^2 + 1$ (2 marks)

a. ii. $|z - i| = |z - u|$

$|x + iy - i| = |x + iy - u|$

$|x + i(y - 1)| = |x - u + iy|$

$\Rightarrow x^2 + (y-1)^2 = (x-u)^2 + y^2$

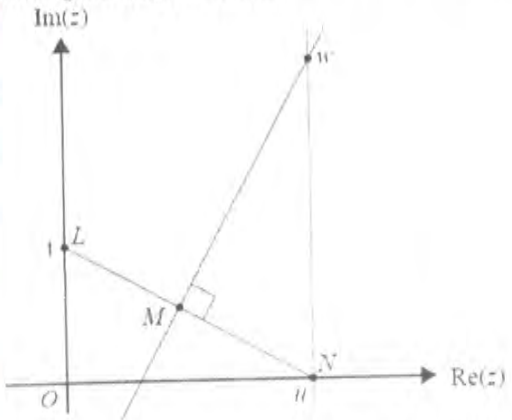
$x^2 + y^2 - 2y + 1 = x^2 - 2xu + u^2 + y^2$

$-2y + 1 = -2ux + u^2$

$2y - 1 = 2ux - u^2$

$2y = 2ux - u^2 + 1$ (Shown)

Let w denote the point of intersection of the perpendicular bisector of LN and the line defined by $\text{Re}(z) = u$ as shown in the Argand diagram below.



b) i) Show that $w = u + \frac{1}{2}(u^2 + 1)i$ (2 marks)

ii) As u moves along the positive Re(z) axis, w moves along a curve. Find the Cartesian equation of this curve and sketch it on the diagram above. (2 marks)

c) Show that the perpendicular bisector of LN is tangent to the curve at w. (3 marks)

b. i. $y = ux - \dfrac{u^2}{2} + \dfrac{1}{2}$

at $x = u$

$y = u^2 - \dfrac{u^2}{2} + \dfrac{1}{2}$

$= \dfrac{1}{2}\left(u^2 + 1\right)$

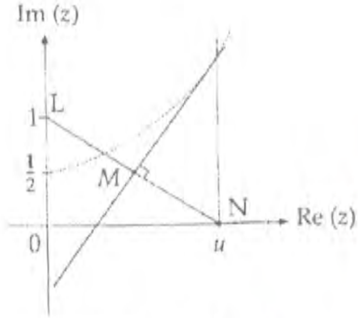
$w = x + yi$

$w = u + \dfrac{1}{2}\left(u^2 + 1\right)i$ (Shown)

b. ii. $w = u + \frac{1}{2}\left(u^2 + 1\right)i$

$x = u \qquad y = \frac{1}{2}(u^2 + 1)$

$y = \frac{1}{2}\left(x^2 + 1\right), \ x > 0$



c. $y = \dfrac{1}{2}(x^2 + 1)$

$\dfrac{dy}{dx} = x$

Tangent at $x = u \quad y = \dfrac{1}{2}(u^2 + 1)$

Given by $y - y_1 = \dfrac{dy}{dx}(x - x_1)$ at $x = u$

$y = \dfrac{1}{2}(u^2 + 1) = u(x - u)$

$2y - u^2 - 1 = 2ux - 2u^2$

$2y = 2ux - u^2 + 1$

From a. ii. we know that this is the equation of the perpendicular bisector of LN therefore the perpendicular bisector is tangent to the curve at w.